Some Geometrical Theorems Connected with Laplace's Equation and the Equation of Wave Motion.

BY H. BATEMAN.

§ 1.

If A_1, A_2 are two points whose coordinates referred to rectangular axes (OX, OY) are $(a + \rho\cos\theta,\ b + \rho\sin\theta)$, $(a - \rho\cos\theta,\ b - \rho\sin\theta)$ respectively, then the points B_1, B_2 whose coordinates are $(a + i\rho\sin\theta,\ b - i\rho\cos\theta)$, $(a - i\rho\sin\theta,\ b + i\rho\cos\theta)$ respectively are called the *anti-points* of A_1 and A_2.* The lines A_1B_2, A_1B_1, A_2B_1, A_2B_2 are isotropic lines and the four points A_1, A_2, B_1, B_2 may be regarded as the four foci of a conic; if A_1 and A_2 are the real foci, then B_1 and B_2 are the imaginary foci. The lines A_1A_2, B_1B_2 are perpendicular and are bisected at their point of intersection $C(a, b)$.

In the following theorems the quantities a, b, ρ, θ need not be real, but the interpretation of the theorems is facilitated when this is the case.

THEOREM I. *If V is a uniform solution of Laplace's equation*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

the sum of the values of V at A_1 and A_2 is equal to the sum of the values of V at B_1 and B_2.

Let

$$V = f(x + iy) + g(x - iy),$$

then

$$\begin{aligned}
V(A_1) &= f(a + ib + \rho e^{i\theta}) + g(a - ib + \rho e^{-i\theta}),\\
V(A_2) &= f(a + ib - \rho e^{i\theta}) + g(a - ib - \rho e^{-i\theta}),\\
V(B_1) &= f(a + ib + \rho e^{i\theta}) + g(a - ib - \rho e^{-i\theta}),\\
V(B_2) &= f(a + ib - \rho e^{i\theta}) + g(a - ib + \rho e^{-i\theta}).
\end{aligned}$$

* These points are also called associated points. See Darboux: "Sur une classe remarquable de courbes et de surfaces algebriques," 2nd ed., p. 61. The term used here is due to Cayley.

Therefore,

$$V(A_1) + V(A_2) = V(B_1) + V(B_2),$$

the symbol $V(A_1)$ being used to denote the value of V at the point A_1.

If in particular we take

$$V = \log \sqrt{(x-\xi)^2 + (y-\eta)^2} \quad \text{or} \quad V = \tan^{-1} \frac{y-\eta}{x-\xi},$$

where (ξ, η) are the coordinates of an arbitrary point P, we obtain the well-known theorems

$$MA_1 \,.\, MA_2 = MB_1 \,.\, MB_2,$$

$$A_1 \hat{M} X + A_2 \hat{M} X = B_1 \hat{M} X + B_2 \hat{M} X.$$

The last theorem indicates that the angles $A_1 M A_2$ and $B_1 M B_2$ have the same bisectors.

THEOREM II. *If* $u + iv = f(x+iy)$, *so that*

$$2u = f(x+iy) + f(x-iy),$$
$$2iv = f(x+iy) - f(x-iy),$$

then the difference of the two values of u *at the points* A_1 *and* A_2 *is equal to the difference of the two values of* $2iv$ *at the points* B_1 *and* B_2.

We have

$$2u(A_1) = f(a+ib+\rho e^{i\theta}) + f(a-ib+\rho e^{-i\theta}),$$
$$2u(A_2) = f(a+ib-\rho e^{i\theta}) + f(a-ib-\rho e^{-i\theta}),$$
$$2iv(B_1) = f(a+ib+\rho e^{i\theta}) - f(a-ib-\rho e^{-i\theta}),$$
$$2iv(B_2) = f(a+ib-\rho e^{i\theta}) - f(a-ib+\rho e^{-i\theta});$$

hence

$$u(A_1) - u(A_2) = iv(B_1) - iv(B_2).$$

If we take

$$u = \log \sqrt{(x-\xi)^2 + (y-\eta)^2},$$
$$v = \tan^{-1} \frac{y-\eta}{x-\xi},$$

we obtain the well-known theorem*

$$\log \frac{MA_1}{MA_2} = i B_1 \hat{M} B_2 .$$

This second theorem is important because, if A_1 and A_2 lie on a curve $u = \text{const.}$, we have $u(A_1) = u(A_2)$, and so it follows that B_1 and B_2 lie on a curve $v = \text{const.}$

* Cf. Darboux, *loc. cit.*, p. 63; Laguerre, *Nouvelles Annales de Mathématique* (1853), p. 7.

This is a theorem of very wide application, and so it may be worth while to consider a few particular cases.

If $x + iy = \cos(u + iv)$, the curves $u = \text{const.}$, $v = \text{const.}$ are confocal conics, and we have the theorem that, if A_1 and A_2 are any two points on a conic, the two anti-points B_1 and B_2 lie on a confocal. If we take a system of parallel chords $A_1 A_2$, it is easy to see that the anti-points B_1, B_2 trace out a confocal, and by varying the direction of the chords different confocals are obtained.

If $x + iy = \text{sn}^2 \tfrac{1}{2}(u + iv)$, the curves $u = \text{const.}$, $v = \text{const.}$ are confocal Cartesian ovals. Hence, if A_1, A_2 are any two points on a Cartesian oval, the anti-points B_1, B_2 lie on a confocal Cartesian oval.

This theorem may be used to obtain a property of the lines of curvature of a quadric surface in the following way:

It is known that, if the points of a quadric are projected from an umbilic on to a plane parallel to the tangent plane at the umbilic, then the lines of curvature project into confocal Cartesian ovals,* and the other three umbilici, which lie on the same focal conic as the vertex of projection, project into three collinear foci of the ovals. Now a quadrilateral $a_1 b_1 a_2 b_2$ formed of generators of the quadric will project into a quadrilateral formed of isotropic lines, for any plane section of the quadric projects into a circle, and so a section consisting of two generators will project into two isotropic lines. The points A_1, A_2, which are the projections of a_1, a_2, will thus be the anti-points of the points B_1, B_2, which are the projections of b_1, b_2. The theorem relating to the points A_1, A_2, B_1, B_2 and the two confocal Cartesian ovals tells us that, if a_1 and a_2 lie on a line of curvature $u = \text{const.}$, the points b_1 and b_2 will lie on a line of curvature $v = \text{const.}$ This theorem may also be stated as follows:

If one pair of opposite corners of a quadrilateral formed of generators of a hyperboloid of one sheet lie on a confocal ellipsoid, the other pair of opposite corners will lie on a confocal hyperboloid of two sheets.

This is really a particular case of a well-known theorem relating to the geodesics on a quadric.

§ 2.

We shall now consider some extensions of Theorems I and II to spaces of three and four dimensions.

* W. K. Clifford, *Educational Times* (1873), Reprint, Vol. XIX, pp. 73, 74. *Math. Papers*, p. 177.

If, in a space of three dimensions, S_3, we take the anti-points of $A_1 A_2$ for all the planes through $A_1 A_2$, the locus of these anti-points B_1, B_2 is a circle, for the locus is obtained by rotating $A_1 B_1 A_2$ about the line $A_1 A_2$. The points A_1, A_2 are called the *foci* of this circle,[*] for they are the centres of isotropic cones passing through the circle.

Conversely, if we are given a circle c, we can determine the vertices of the two isotropic cones which pass through the circle, and call these the foci of the circle. If the circle is of radius ρ and its centre is at a, b, c, and the direction cosines of its plane are (l, m, n), then the coordinates of the two foci are

$$(a \pm i\rho l, \quad b \pm i\rho m, \quad c \pm i\rho n).$$

It is clear that, if the circle is real, the coordinates of the foci are complex quantities. This fact has been used by Chasles[†] and Laguerre[‡] to obtain a geometrical representation of points with complex coordinates, a point being represented by a real circle. To make the correspondence unique Laguerre introduces the idea of a cycle; *i. e.*, a circle described in a definite direction; one focus is then represented by one cycle and the other focus by the opposite cycle.

In a space of four dimensions, S_4, the anti-points to $A_1 A_2$, which lie in planes through $A_1 A_2$, all lie on a sphere centre C; the linear space S_3 containing this sphere is perpendicular to the line $A_1 A_2$; and the points A_1, A_2 are the vertices of the isotropic cones containing the sphere. The points A_1, A_2 are called the *foci* of the sphere.

If we use (x_1, x_2, x_3, x_4) to denote the coordinates of a point in S_4 when rectangular axes are used, then the coordinates of the foci of a sphere of radius ρ having its centre at the point (c_1, c_2, c_3, c_4) are

$$c_1 \pm i\rho\lambda_1, \quad c_2 \pm i\rho\lambda_2, \quad c_3 + i\rho\lambda_3, \quad c_4 \pm i\rho\lambda_4,$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the direction cosines of the line joining the two foci. These quantities are connected by the equation

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1,$$

and the equation of the space containing the sphere is

$$\lambda_1 (x_1 - c_1) + \lambda_2 (x_2 - c_2) + \lambda_3 (x_3 - c_3) + \lambda_4 (x_4 - c_4) = 0.$$

---

[*] Darboux, *Annales de l'École Normale*, t. 1 (1872), 2nd series, p. 323.

[†] "Géometrie supérieure," pp. 546–566; "Théorie des coniques sphériques homofocales," *Liouville's Journal*, t. 5 (1860), 2nd series, p. 426. See also Cayley, *Annali di Matematica*, t. 1.

[‡] *Bulletin de la Société Philomatique* (1870); *Nouvelles Annales de Mathématiques* (1872).

The line whose equations are

$$\frac{x_1 - a_1}{l_1} = \frac{x_2 - a_2}{l_2} = \frac{x_3 - a_3}{l_3} = \frac{x_4 - a_4}{l_4}$$

is said to be *isotropic* when $l_1^2 + l_2^2 + l_3^2 + l_4^2 = 0$. The isotropic lines which pass through the point (a_1, a_2, a_3, a_4) all lie on the *isotropic cone* or point-hypersphere, whose equation is

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 + (x_4 - a_4)^2 = 0.$$

The two isotropic cones whose equations are

$$(x_1 - c_1 - i\rho\lambda_1)^2 + (x_2 - c_2 - i\rho\lambda_2)^2 + (x_3 - c_3 - i\rho\lambda_3)^2 + (x_4 - c_4 - i\rho\lambda_4)^2 = 0,$$
$$(x_1 - c_1 + i\rho\lambda_1)^2 + (x_2 - c_2 + i\rho\lambda_2)^2 + (x_3 - c_3 + i\rho\lambda_3)^2 + (x_4 - c_4 + i\rho\lambda_4)^2 = 0$$

intersect in the sphere whose equations are

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \rho^2,$$
$$\lambda_1 (x_1 - c_1) + \lambda_2 (x_2 - c_2) + \lambda_3 (x_3 - c_3) + \lambda_4 (x_4 - c_4) = 0;$$

and this is the sphere considered previously.

The sphere is evidently real when $c_1, c_2, c_3, c_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and ρ are all real, and in this case the coordinates of the two foci are conjugate complex quantities.

A real sphere may be used to represent a point in S_4 with complex coordinates, just as a real circle was used to represent points in S_3. This representation is of special interest when the sphere lies in the space $x_4 = 0$ and will be discussed later.

If a sphere K passes through a fixed circle γ, its foci will always lie on a second fixed circle $\bar{\gamma}$, which is the locus of the foci of γ for different spaces containing γ. The circles γ and $\bar{\gamma}$ are evidently concentric and lie in perpendicular planes. The relation between the two circles is a mutual one and each is the locus of the vertex of an isotropic cone passing through the other. We shall say that each circle is the focal circle of the other.

If we project two such circles γ and $\bar{\gamma}$ orthgonally onto a space S_3, we obtain two conics, Γ and $\bar{\Gamma}$, which are focal conics of a system of confocal quadrics.

The focal properties of quadrics may in fact be developed very conveniently in connection with the focal properties of circles; thus we have the following theorem:

The foci of a set of circular sections of a quadric lie on a focal conic. Conversely, if we take the circles whose foci are the extremities of a system of parallel chords of a conic, these circles will generate a quadric having the conic as a focal conic. By varying the direction of the chords a system of confocal quadrics is obtained.

If we call the other focal conics of a quadric having Γ as focal conic, the focals of Γ, we have the following theorem:*

If Γ *lies on a quadric* Q *and* $\overline{\Gamma}$ *is a focal of* Γ, *then* $\overline{\Gamma}$ *lies on a quadric confocal to* Q

These theorems are probably well-known,† but they are not given in the usual text-books; so we shall proceed to prove them.

Let $Q=0$, $\Omega=0$ be the tangential equations of a quadric and of the circle at infinity respectively. Let $C=0$ be the tangential equation of the conic cut out on Q by a plane t, and let $T=0$ be the tangential equation of the pole of t; then there is an identity of the form

$$Q \equiv C - \lambda T^2.$$

Now let $\Gamma=0$ be the tangential equation of a focal conic of C; then we have an identity of the form

$$\Gamma = C + \mu \Omega.$$

These two equations give

$$Q + \mu \Omega = \Gamma - \lambda T^2.$$

This identity shows that Γ is a plane section of a quadric $Q+\mu\Omega=0$, which is confocal with Q. If C is a circle, Γ may consist of the two foci of the circle and of the line joining them, and this line will lie in one of the principal planes of Q on account of the position of a circular section of Q. Also the point T lies in this principal plane; hence, the equation

$$Q + \mu \Omega = \Gamma - \lambda T^2$$

represents a conic confocal with Q, and so the two foci of C lie on a focal conic of Q.

Now let the point T move along a diameter of Q; then the conics C will be cut out by a family of parallel planes. If C' is a conic cut out by the polar

* Some analogous theorems have been given by Chasles, *Liouville's Journal*, t. 5 (1860), pp. 444–454, and Darboux, "Sur les théorèms d'Ivory, relatifs aux surfaces homofocales du second degré," Paris, Gauthier-Villars (1872).

† Most of them are given by Humbert, *Bull. de la Soc. Math. de France*, t. 13 (1885), p. 95.

plane of T' and

$$Q = C' - \lambda' T'^2,$$
$$\Gamma' = C' + \mu' \Omega,$$

we shall have

$$\Gamma' - \Gamma = C' - C + (\mu' - \mu)\,\Omega = \lambda' T'^2 - \lambda T^2 + (\mu' - \mu)\,\Omega.$$

Now, since C' is similar and similarly situated to C, Γ' will also be similar and similarly situated to Γ. Also the centres of Γ and Γ' lie on TT', and the planes of the two conics are parallel; hence, the cones containing Γ and Γ' will have their vertices on the line TT' and will touch the quadric $\Gamma' - \Gamma = 0$.

On the other hand, the equation

$$\lambda' T'^2 - \lambda T^2 + (\mu' - \mu)\,\Omega = 0$$

either represents a surface of revolution having the points T, T' as foci and the line TT' as axis, or, if $\mu' = \mu$, it represents a pair of points.

In the first case the tangent cone from a point on TT' would be a cone of revolution, and so the two cones containing the conics Γ and Γ' would be cones of revolution. This, however, is only the case when these conics are circles having their planes perpendicular to TT', and this is not true in the general case under consideration. Consequently, we must have $\mu' = \mu$, and this implies that the conic Γ' lies on the same confocal as Γ. Hence, we have the theorem that when the conics C are parallel sections of Q, the conics Γ are parallel sections of a confocal quadric $Q + \mu\,\Omega$. In the particular case when the conics C are circular sections of Q, the conics Γ will be pairs of points which are extremities of a system of parallel chords of a focal conic of Q.

We have paid special attention to these theorems because in a sense they provide us with an analogue of Theorem II, when we take into account the fact that a system of confocal ellipsoids are equipotential surfaces and the confocal hyperboloids their orthogonal trajectories. The ellipsoids may thus be supposed to correspond to the loci $u = \text{const.}$ and the confocal hyperboloids to the loci $v = \text{const.}$ The pair of points on $u = \text{const.}$ and the pair of anti-points on $v = \text{const.}$ are now replaced by a conic on $u = \text{const.}$ and a focal conic on $v = \text{const.}$

It looks as if there should be analogous theorems for other equipotential surfaces, but we shall not investigate the matter here. We shall now pass on to some analogues of the first theorem.

THEOREM III. *If V is a solution of the partial differential equation*

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}=0,$$

the mean value of V over a sphere in S_4 is equal to the mean value of V along the line joining the foci.

This theorem is practically proved in the usual derivation of Poisson's formula for a solution of the equation of wave motion.*

To prove the theorem we observe in the first place that the differential equation is unaltered by a change of rectangular axes; hence we may choose the axes in such a way that the equations of the sphere are

$$(X_1-x_1)^2+(X_2-x_2)^2+(X_3-x_3)^2=\rho^2, \quad X_4=x_4,$$

where X_1, X_2, X_3, X_4 are current coordinates. We have then to establish the identity of the two integrals

$$W_1=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} F(x_1+\rho\sin\theta\cos\phi,\ x_2+\rho\sin\theta\sin\phi,\ x_3+\rho\cos\theta,\ x_4)\sin\theta\, d\theta\, d\phi,$$

$$W_2=\frac{1}{2\rho}\int_{-\rho}^{+\rho} F(x_1,\ x_2,\ x_3,\ x_4+iu)\,du,$$

where $V=F(x_1, x_2, x_3, x_4)$ is a solution of the partial differential equation.

We shall assume that the function F possesses second derivatives which are finite and continuous over the domains of integration, when the variables x_1, x_2, x_3, x_4 are limited to a certain domain. The integrals W_1 and W_2 may then be differentiated by the rule of Leibnitz, and it is easy to verify that they satisfy the partial differential equations

$$\frac{\partial^2 W}{\partial x_1^2}+\frac{\partial^2 W}{\partial x_2^2}+\frac{\partial^2 W}{\partial x_3^2}+\frac{\partial^2 W}{\partial x_4^2}=0,$$

$$\frac{\partial^2 W}{\partial x_1^2}+\frac{\partial^2 W}{\partial x_2^2}+\frac{\partial^2 W}{\partial x_3^2}=\frac{\partial^2 W}{\partial \rho^2}-\frac{2}{\rho}\frac{\partial W}{\partial \rho},$$

$$\frac{\partial^2 W}{\partial x_4^2}+\frac{\partial^2 W}{\partial \rho^2}-\frac{2}{\rho}\frac{\partial W}{\partial \rho}=0.$$

Now let $F(x_1, x_2, x_3, x_4)$ be a regular function of x_4 within some circle of radius $R>\rho$ surrounding the point x_4; then it can be proved that there is only one solution of the last equation which reduces to $F(x_1, x_2, x_3, x_4)$ when $\rho=0$, and can be expanded in a convergent power-series of the form

$$W=A_0+A_1\rho+A_2\rho^2+A_3\rho^3+\dots\dots$$

* See for instance Rayleigh's *Sound*, Vol. II, p. 99.

For, when this power-series is substituted in the differential equation, we obtain the relations

$$0 = A_1 = A_3 = A_5 = \ldots .$$

and a recurrence formula by which the coefficients $A_2, A_4, \ldots .$ may be calculated in a unique manner from the first coefficient A_0, which is equal to $F(x_1, x_2, x_3, x_4)$.

Now, both W_1 and W_2 are equal to $F(x_1, x_2, x_3, x_4)$ when $\rho = 0$; hence, since they both satisfy the differential equation, we must have $W_1 = W_2$. This gives the formula

$$\int_{-\rho}^{+\rho} F(x_1, x_2, x_3, x_4 + iu)\, du = \frac{\rho}{2\pi} \int_0^\pi \int_0^{2\pi} F(x_1 + \rho \sin\theta \cos\phi, \\ x_2 + \rho \sin\theta \sin\phi,\ x_3 + \rho\cos\theta,\ x_4) \sin\theta\, d\theta\, d\phi.$$

To derive Poisson's integral from this formula, we express the integral from $-\rho$ to $\rho + 2\varepsilon$ by means of an integral taken over a consecutive sphere, thus:

$$\int_{-\rho}^{\rho+2\varepsilon} F(x_1, x_2, x_3, x_4 + iu)\, du = \frac{\rho + \varepsilon}{2\pi} \int_0^\pi \int_0^{2\pi} F(x_1 + \overline{\rho + \varepsilon} \sin\theta \cos\phi, \\ x_2 + \overline{\rho + \varepsilon} \sin\theta \sin\phi,\ x_3 + \overline{\rho + \varepsilon} \cos\theta,\ x_4 + i\varepsilon) \sin\theta\, d\theta\, d\phi.$$

Subtracting, we get

$$F(x_1, x_2, x_3, x_4 + i\rho) = \frac{1}{4\pi} \frac{d}{d\rho} \rho \int_0^\pi \int_0 \quad F(x_1 + \rho \sin\theta \cos\phi, \\ x_2 + \rho \sin\theta \sin\phi,\ x_3 + \rho\cos\theta,\ x_4) \sin\theta\, d\theta\, d\phi \\ + \frac{i\rho}{4\pi} \frac{\partial}{\partial x_4} \int_0^\pi \int_0^{2\pi} F(x_1 + \rho\sin\theta\cos\phi,\ x_2 + \rho \sin\theta \sin\phi, \\ x_3 + \rho\cos\theta,\ x_4) \sin\theta\, d\theta\, d\phi.$$

Putting

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = 0, \quad \rho = t,$$
$$F(x, y, z, 0) = f(x, y, z),$$
$$\left[\frac{\partial}{\partial t} F(x, y, z, it)\right]_{t=0} = g(x, y, z),$$

we obtain the equation

$$F(x, y, z, t) = \frac{1}{4\pi} \frac{d}{dt} t \int_0^\pi \int_0^{2\pi} f(x_1 + t \sin\theta\cos\phi, \\ x_2 + t\sin\theta\sin\phi,\ x_3 + t\cos\theta) \sin\theta\, d\theta\, d\phi \\ + \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} g(x_1 + t\sin\theta\cos\phi,\ x_2 + t \sin\theta\sin\phi, \\ x_3 + t\cos\theta) \sin\theta\, d\theta\, d\phi.$$

This is Poisson's formula.

THEOREM IV. *If* $V = F(x_1, x_2, x_3, x_4)$ *is a solution of*

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} + \frac{\partial^2 V}{\partial x_4^2} = 0,$$

the mean value of V *round a circle in* S_4 *is equal to* i *times the mean value of* V *round the focal circle.*

As before, we can transform the axes so that the equations of the two circles take a simple form such as

$$\begin{aligned} x_1^2 + x_2^2 &= \rho^2, \qquad x_3 = x_4 = 0, \\ x_3^2 + x_4^2 &= -\rho^2, \qquad x_1 = x_2 = 0. \end{aligned}$$

We now have to establish the identity of the two integrals

$$U_1 = \frac{1}{2\pi}\int_0^{2\pi} F[x_1 + \rho\cos\alpha,\ x_2 + \rho\sin\alpha,\ x_3,\ x_4]\, d\alpha,$$

$$U_2 = \frac{1}{2\pi}\int_0^{2\pi} F[x_1,\ x_2,\ x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\, d\theta.$$

These integrals both satisfy the differential equations

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = 0,$$

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho},$$

$$\frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} + \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} = 0,$$

and reduce to $F(x_1, x_2, x_3, x_4)$ when $\rho = 0$.

Now, if F is a regular function of x_3 and x_4 in circles of radii $R > \rho$ surrounding the points x_3 and x_4, in their complex planes, it may be shown as before that there is only one solution of the last differential equation which reduces to $F(x_1, x_2, x_3, x_4)$ when $\rho = 0$. Hence, we must have $U_1 = U_2$ and so

$$\int_0^{2\pi} F[x_1 + \rho\cos\alpha,\ x_2 + \rho\sin\alpha,\ x_3,\ x_4]\, d\alpha$$
$$= \int_0^{2\pi} F[x_1,\ x_2,\ x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\, d\theta.$$

We now proceed to some applications of this result.

(1) Let F have the same value at all points of a circle

$$(X_1 - x_1)^2 + (X_2 - x_2)^2 = \rho^2, \qquad X_3 = 0, \qquad X_4 = 0;$$

then this value $F\{x_1, x_2, x_3, x_4, \rho\}$ is given by the formula

$$F\{0, 0, x_3, x_4, \rho\} = \frac{1}{2\pi}\int_0^{2\pi} F[0, 0, x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\,d\theta.$$

If, then, we know the value of F at points of the plane $X_1 = x_1$, $X_2 = x_2$, its value at other points of space can be derived from the formula. The formula may be written in the more convenient form

$$V = F\{x_3, x_4, \rho\} = \frac{1}{2\pi}\int_0^{2\pi} f[x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\,d\theta,$$

where $f(x_3, x_4)$ is the value of V when $\rho = 0$. If V is independent of x_4, and therefore a solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} = 0,$$

the general formula becomes

$$\int_0^{2\pi} F[x_1 + \rho\cos\alpha,\ x_2 + \rho\sin\alpha,\ x_3]\,d\alpha = \int_0^{2\pi} F[x_1, x_2, x_3 + i\rho\cos\theta]\,d\theta.$$

This equation expresses that the mean value of V round a circle is equal to the integral of V multiplied by a certain function along the line joining the foci of the circle. The formula may be used to obtain the well-known formula

$$V = \frac{1}{2\pi}\int_0^{2\pi} F[x_3 + i\rho\cos\theta]\,d\theta$$

for a solution which is symmetrical about the axis of x_3. A more general result may be obtained by using the integrals along a pair of focal circles situated arbitrarily in S_4. Thus we have the relation

$$\int_0^{2\pi} F(\xi_1 + \rho\cos\alpha\cos\omega,\ \xi_2 + \rho\sin\alpha,\ \xi_3 + \rho\cos\alpha\sin\omega,\ \xi_4)\,d\alpha$$
$$= \int_0^{2\pi} F(\xi_1 - i\rho\cos\theta\sin\omega,\ \xi_2,\ \xi_3 + i\rho\cos\theta\cos\omega,\ \xi_4 + i\rho\sin\theta)\,d\theta,$$

when $V = F(\xi_1, \xi_2, \xi_3, \xi_4)$ is a suitable solution of

$$\frac{\partial^2 V}{\partial \xi_1^2} + \frac{\partial^2 V}{\partial \xi_2^2} + \frac{\partial^2 V}{\partial \xi_3^2} + \frac{\partial^2 V}{\partial \xi_4^2} = 0.$$

When V is independent of ξ_1 and consequently a solution of

$$\frac{\partial^2 V}{\partial \xi_2^2} + \frac{\partial^2 V}{\partial \xi_3^2} + \frac{\partial^2 V}{\partial \xi_4^2} = 0,$$

this equation becomes

$$\int_0^{2\pi} F(x + \rho \sin\alpha,\ y + \rho \cos\alpha \sin\omega,\ z)\, d\alpha$$
$$= \int_0^{2\pi} F(x,\ y + i\rho \cos\theta \cos\omega,\ z + i\rho \sin\theta)\, d\theta,$$

where x, y, z are written in place of ξ_2, ξ_3, x_4.

This equation indicates that an integral of a solution of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

taken round the ellipse

$$\frac{(X - x)^2}{\rho^2} + \frac{(Y - y)^2}{\rho^2 \sin^2\omega} = 1, \quad Z = z,$$

is equal to an integral of V taken over a portion of the focal conic

$$\frac{(Y - y)^2}{\rho^2 \cos^2\omega} + \frac{(Z - z)^2}{\rho^2} + 1 = 0, \quad X = x.$$

If we take an imaginary value of ω so that $\sin^2\omega > 1$, $\cos^2\omega < 0$, the focal conic is the focal hyperbola of the ellipse.

The theorem may be used in potential theory in the following way: If a solution of Laplace's equation is known to be constant at all real and imaginary points of an ellipse or hyperbola, then its value on this conic may be found when the values of the function at points on a focal conic are known. The theorem may be regarded as an analogue of Theorem I.

§ 3.

We have already remarked that a sphere in the space $x_4 = 0$ may be used to represent a point in S_4 whose coordinate x_4 is imaginary. This representation has been used with great success by Darboux* in investigating properties of systems of spheres. We shall use it here in connection with a second representation, in which a particle which is at the point (x, y, z) at time t is represented by a point in S_4 whose coordinates are (x_1, x_2, x_3, x_4), where

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict,$$

and c is the velocity of light.

---

* *Annales de l'École Normale*, t. 1 (1872), 2nd ser., p. 323. The points on a hypersphere in S_4 correspond to spheres which cut a fixed sphere at a constant angle, the points on a sphere in S_4 to a system of spheres touching two fixed spheres and the points on a circle in S_4 to spheres touching three fixed spheres.

This transformation arises naturally in the reduction of the equation of wave motion,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

to the symmetrical form

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} + \frac{\partial^2 V}{\partial x_4^2} = 0.$$

It has been used recently by Minkowski* in connection with the equations of electrodynamics and the transformations which form the basis of the theory of relativity. The transformation reduces the equations of electrodynamics to a symmetrical form, and so enables us to use mathematical symmetry as a guide to further investigations.

When we combine the representations of Darboux and Minkowski, we are led to a representation of a space-time point (*i. e.*, the instantaneous position of a particle at a given time) by means of a sphere with a definite sign attached to its radius. It is convenient to regard the sphere as a spherical wave; the sign of its radius is then indicated by the direction in which the wave is moving. A space-time point (x, y, z, t) is represented by a spherical wave whose centre is at the point (x, y, z) and whose radius is equal to ct, the wave being a convergent wave, if t is positive, and a divergent wave when t is negative. The wave may be regarded as the position at time $t = 0$ of a disturbance which will be absorbed by (or has been emitted by) a particle which is at (x, y, z) at time t.

We shall call the wave the representative wave of the particle at time t. It is clear that the career of a particle may be studied very conveniently with the aid of its representative waves at different times. If the particle is moving with a velocity less than that of light, no two of the convergent representative waves can intersect and similarly no two of the divergent representative waves can intersect.

It follows from this result and the principle of Huyghens that there is only one position of the particle in which it can create a disturbance which will arrive at the point (x, y, z) at time t. For, if a disturbance issuing from a point (x_0, y_0, z_0) at time t_0 is propagated in the form of a spherical wave with this point as centre, then the wave will only reach the point (x, y, z) at time t, if

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2 (t - t_0)^2.$$

* *Göttinger Nachrichten* (1908).

The geometrical interpretation of this equation is that the representative wave of (x, y, z, t) is touched internally by the representative wave of (x_0, y_0, z_0, t_0), if both waves are convergent (or divergent), and touched externally by it, if one wave is convergent and the other one divergent.

Now, since no two convergent waves associated with the particle intersect, there is only one wave of the system which has the proper type of contact with the representative wave of the point (x, y, z, t).

This theorem, which is proved analytically by Prof. A. W. Conway,* is of considerable importance in the mathematical theory of the motion of an electron. If the electron be treated as a simple singularity of an electromagnetic field, Liénard and Wiechert have shown that an appropriate electromagnetic field for an electron moving in an arbitrary manner may be obtained in the following way:

Let the motion of the electron or singularity be specified by the equations

$$\xi = \xi(\tau), \quad \eta = \eta(\tau), \quad \zeta = \zeta(\tau);$$

then there is one value of $\tau < t$ which satisfies the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2(t - \tau)^2.$$

If, with this value of τ,

$$M = 2c^2(t - \tau) - 2\xi'(\tau)[x - \xi(\tau)] - 2\eta'(\tau)[y - \eta(\tau)] - 2\zeta'(\tau)[z - \zeta(\tau)],$$

the components (E_x, E_y, E_z), (H_x, H_y, H_z) of the electric and magnetic forces at the point (x, y, z) at time t are given by the equations

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}, \qquad H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \text{ etc.,}$$

where

$$A_x = e\frac{\xi'(\tau)}{M}, \quad A_y = e\frac{\eta'(\tau)}{M}, \quad A_z = e\frac{\zeta'(\tau)}{M}, \quad \Phi = \frac{ec}{M}.$$

These potentials A_x, A_y, A_z, Φ are all solutions of the equation of wave motion,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

and in fact this equation is satisfied by any function of the form

$$V = \frac{1}{M} f(\tau).$$

These expressions for the components of the electric and magnetic forces show that the field at (x, y, z, t) depends only on the position τ of the moving

---

* *Proc. London Math. Soc.*, Vol. I (1903), Ser. 2, p. 154.

singularity. Lorentz* calls this position of the electron the *effective position.* Minkowski† calls it the "light point" of (x, y, z, t). It is convenient, however, to have a term which will apply to an electron irrespective of its path, and so we shall say that an electron (or particle) $P(\xi, \eta, \zeta, \tau)$ is *active* relative to $Q(x, y, z, t)$ when the conditions

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = c^2(t-\tau)^2, \quad t > \tau \tag{1}$$

are satisfied. Thus P is active relative to Q whenever a convergent representative wave of Q is touched internally by a convergent representative wave of P, or in fact, whenever the two representative waves touch in the proper way.

If we represent P, Q by points $(\xi_1, \xi_2, \xi_3, \xi_4)$, (x_1, x_2, x_3, x_4) in S_4, where

$$\xi_1 = \xi, \quad \xi_2 = \eta, \quad \xi_3 = \zeta, \quad \xi_4 = ic\tau,$$
$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict,$$

then the condition (1) takes the form

$$(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 + (x_4 - \xi_4)^2 = 0,$$

and this implies that the line joining the two representative points in S_4 is an *isotropic line.*

Hence, if P is active relative to Q (or vice-versa), the representative points in S_4 lie on an isotropic line. In studying different configurations of electrons we may either consider the positions of the different electrons at a given time, or we may consider the positions in which they are active relative to one another. The second method is preferable for some purposes, for the relation between the electrons is unaltered by changing the axes from a system at rest to a system in uniform motion. We shall therefore proceed to investigate some properties of isotropic lines in S_4, for the properties of systems of electrons which are actively related to one another can be expressed in terms of these.

§ 4.

Since the equation of an isotropic line is of the form

$$(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 + (x_4 - \xi_4)^2 = 0,$$

it is clear that an isotropic line is transformed into an isotropic line by any transformation which leaves this condition unaltered in form. The group of conformal transformations of the space S_4 satisfies this requirement and will be

---

* "The Theory of Electrons," Leipzig (1909), p. 251.

† *Physikalische Zeitschrift* (1909).

referred to as the group G_{15}.* The equivalent group of transformations in the variables (x, y, z, t) leaves the equation

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2$$

unaltered in form and so transforms a spherical wave into a spherical wave.† The transformations of the group have on this account been called *spherical wave transformations*; they are characterized by a relation of the form

$$dx'^2+dy'^2+dz'^2-c^2dt'^2=\lambda^2[dx^2+dy^2+dz^2-c^2dt^2].$$

It has been proved elsewhere‡ that the fundamental equations of the theory of electrons are covariant for transformations of this type. The equation of

---

* It is a fifteen-parameter group. The group of Lorentzian transformations, considered by Poincaré, *Rend. Palermo* (1906), and Minkowski, *Gött. Nachr.* (1908), is a subgroup of G_{15}; it is only a six-parameter group and is included in the ten-parameter group of transformations of rectangular axes in S_4.

† Einstein's postulate of the constancy of the velocity of light (*Ann. d. Physik*, Vol. XVII, 1905) is really equivalent to the assumption that there is a group of transformations for which the equations

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2 \qquad t>\tau \tag{1}$$

are covariant. If we regard these as the conditions of observation or for the action of one particle on another, then the corresponding group of transformations, for which these conditions are covariant may be regarded as the *principal group* underlying all physical phenomena. According to the general principles of group-theory, the quantities and relations which are invariant with regard to the principal group should represent true physical quantities and relations. Some of these invariants for the group G_{15} have been found by Einstein, Poincaré, Minkowski, Planck, Cunningham and the author. It is desirable that all the principal invariants and covariants for the group should be found, for then we shall perhaps be able to decide whether Einstein's conditions of observation are the right ones or not.

At first sight it would look as if these conditions of observation were only valid for the free æther, since light and other disturbances travel with different velocities in different material media. If, however, we adopt Lorentz's hypothesis that the fundamental equations of electromagnetism for ponderable bodies can be derived from the electron equations by a process of averaging; then we may assume that the condition of observation holds for two neighboring particles, when there is only free æther in between. The conditions of action of two *neighboring* particles are then

$$dx^2+dy^2+dz^2=c^2dt^2 \qquad dt>0. \tag{2}$$

Now, it is a remarkable fact that the group of transformations for which these conditions are covariant is exactly the same as that for which the more general conditions

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2 \qquad t>\tau \tag{1}$$

are covariant. Hence, if we may assume that (2) is the condition for contact action and define the principal group by means of this condition, then we may conclude that the condition for action at a distance is covariant for the principal group whenever the condition can be expressed in the form (1). This result may be expressed in another way by saying that any transformation of the group G transforms free æther into free æther.

The conditions of contact action expressed by equations (2) are the most suitable for the discussion of wave theory in isotropic media, but it is useful to know what difference it would make in the general equations of the electromagnetic field if the condition of contact action were of a more general character. This question has been considered by the author, *Proc. London Math. Soc.*, Vol. VIII (1910), Ser. 2, p. 255.

‡ *Proc. London Math. Soc.*, Vol. VIII (1910), Ser. 2, p. 223.

wave motion is also covariant in the sense that if V is a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=0$$

and

$$V ds = V' ds',$$

where

$$ds^2 = dx^2+dy^2+dz^2-c^2dt^2 = dx_1^2+dx_2^2+dx_3^2+dx_4^2,$$
$$ds'^2 = dx'^2+dy'^2+dz'^2-c^2dt'^2 = dx_1'^2+dx_2'^2+dx_3'^2+dx_4'^2,$$

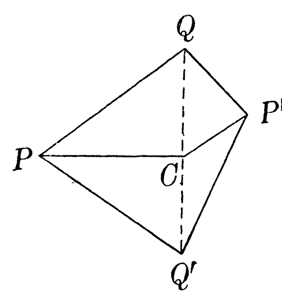
then V' is a solution of

$$\frac{\partial^2 V'}{\partial x'^2}+\frac{\partial^2 V'}{\partial y'^2}+\frac{\partial^2 V'}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2 V'}{\partial t'^2}=0.$$

The last theorem* shows that $\int V ds$ taken along any curve in S_4 is an invariant for transformations of the group G. On this account there is reason to suppose that, when V is properly chosen, the integral represents an important physical quantity. The integral evidently vanishes along an isotropic line, since $ds=0$, and this again brings isotropic lines into prominence. It is interesting to consider the integral to be taken round a closed path, and so we shall consider the properties of closed polygons formed of isotropic lines.

THEOREM V. *If a skew quadrilateral in S_4 is formed of isotropic lines, the four lines lie on a sphere.*

Let $PQP'Q'$ be the corners of the quadrilateral, PQ, QP', $P'Q'$, $Q'P$ the isotropic lines. It is clear that P is one of the anti-points of Q, Q' for the plane PQQ'; hence, if C is the middle point of QQ', PC will be perpendicular to QQ' and $CP=iCQ$. Similarly, CP' is perpendicular to QQ' and $CP'=iCQ=CP$. Now since this is the case, it is possible to draw a sphere to touch the planes QPQ', $QP'Q'$ at P and P', and this sphere will contain the isotropic lines PQ, QP', $P'Q'$, $Q'P$, because a sphere which touches a plane at a point P has for generators the two isotropic lines through P which lie in this plane.



* This theorem may be derived immediately from the analysis given in a paper "On the Conformal Transformations of a Space of Four Dimensions," *Proc. London Math. Soc.*, Vol. VII (1909), Ser. 2, p. 70. There is a more general invariant of type $\int V ds \cos \varepsilon$, where ε is the angle between the path and a member of a given family of space-time curves.

Now let R, R' be the two foci of this sphere, then the lines RP, RP', RQ, RQ', $R'P$, $R'P'$, $R'Q$, $R'Q'$ are isotropic lines, and we have a remarkable figure formed of three pairs of points, PP', QQ', RR', in which the twelve lines joining points belonging to different pairs are all isotropic. The twelve lines also lie by fours on three spheres.

The figure which is derived from this by Darboux's transformation is very interesting. The points PP', QQ', RR' correspond to pairs of waves pp', qq', rr', such that the waves of one pair are in contact with the waves of another pair. The point of contact of the waves p, q is the point in which the space $x_4 = 0$ is met by the line PQ, and similarly for the points of contact of the other sphere. Since the lines PQ, QP', $P'Q'$, $Q'P$ lie on a sphere, it follows that the four points of contact of the spheres pp' with the spheres qq' lie on a circle. Similarly, the points of contact of qq' with rr' lie on a circle, and the points of contact of r and r' with p and p' lie on a circle.

If the waves p, p', etc., are representative waves of particles (p), (p'), etc., at their centres, the relations between the particles may be expressed as follows:

(q), (r), (q'), (r')	are	active	relative	to	(p),
(r), (r'), (p')	"	"	"	"	(q),
(q'), (p')	"	"	"	"	(r),
(q'), (p')	"	"	"	"	(r'),
(p')	is	"	"	"	(q').

An interesting extension of this theorem occurs in Steiner's porism relating to a ring of spheres touching two fixed spheres and having their centres in one plane. This theorem is so well-known that it will not be necessary to give an account of it here.*

Another extension of the theorem may be made in connection with spaces of a higher number of dimensions.

If the sides of a skew hexagon in a space of five dimensions S_5 are isotropic lines, the six sides lie on a hyperspherical locus K_4.

To prove this we consider the section of the figure by any linear space S_4. Any corner of the hexagon and its image in S_4 may be regarded as the foci of a hypersphere in S_4. The hyperspheres which correspond to two corners A_1, A_2

* The following references may prove useful: Steiner, *Werke*, Bd. 1, pp. 135, 455; Geiser, *Einleitung in die synthet. Geometrie* (1869); W. Fiedler, *Cyklographie* (1882), p. 233; Holzmüller, *Elem. d. Stereometrie*, Bd. 1, p. 279; A. Schumann, *Berlin, Askanisches Gymn.* (1883); Maennchen, *Grunert's Archiv* (3), 7, p. 232; J. Finsterbusch, *Atti del. IV Congress dei Matematici*, Vol. I, p. 285.

of the hexagon will touch at the point B_{12} in which the line $A_1 A_2$ meets S_4. Now, if six hyperspheres in S_4 touch one another in succession and the contacts are of the proper kind, the six points of contact lie on a hypersphere.* Hence, the six sides of the hexagon are met by any linear space S_4 in six points lying in a hypersphere K_3, and so the six sides must lie on a hypersphere K_4.

To prove the theorem relating to the six hyperspheres in S_4 we invert the space S_4 into a hypersphere Σ_4. The points of contact of the hyperspheres then invert into the points of contact of the sides of a skew hexagon whose sides all touch Σ_4, and the hyperspheres in S_4 invert into the hyperspheres cut out on Σ_4 by the polars of the vertices of the hexagon. Now the points of contact of the sides of the hexagon lie on a linear space L_4, for if A_1, A_2, A_3, A_4, A_5, A_6 are the corners of the hexagon, B_{12}, B_{23}, B_{34}, B_{45}, B_{56}, B_{61} the points of contact of the sides, we have relations of the type

$$A_1 B_{12} = A_1 B_{61}$$

which lead to the equation

$$\begin{aligned} \pm A_1 B_{12} \,.\, A_2 B_{23} \,.\, A_3 B_{34} \,.\, A_4 B_{45} \,.\, A_5 B_{56} \,.\, A_6 B_{61} \\ = A_2 B_{12} \,.\, A_3 B_{23} \,.\, A_4 B_{34} \,.\, A_5 B_{45} \,.\, A_6 B_{56} \,.\, A_1 B_{61}, \end{aligned} \tag{2}$$

and when the positive sign is taken, this is the condition that the six points of contact should lie in a linear space L_4.

This linear space L_4 will cut Σ_4 in a hypersphere $\overline{K}_3$ containing the six points of contact, and this inverts into a hypersphere K_3 in S_4 containing the six points of contact of the hyperspheres. It should be remarked that the contacts between the hyperspheres which correspond to the corners of a skew hexagon of isotropic lines, are of such a nature that the sign to be taken in equation (2) is always positive; hence, we have the theorem:

THEOREM VI. *If a skew hexagon in S_5 is formed of isotropic lines, its sides lie on a hypersphere K_4.*

There is a corresponding theorem for a polygon of $n+1$ sides in a space of n dimensions ($n > 2$). The case of a skew pentagon in a space of four dimensions may be deduced from Theorem VI by making two of the corners of the hexagon coincident. Since a linear space S_4 can be drawn through five arbitrary points, the sides A_1, A_2, A_3, A_4, A_5 will lie in a liner space S_4, and if A_6 is coincident with A_1, the line $A_1 A_6$ can be taken to be an arbitrary generator of the isotropic

* *Educational Times* (2), 11, Question 16009, pp. 57–61. A solution different from the above was given by D. M. Y. Sommerville.

cone having its vertex at A_1. The theorem then tells us that there are an infinite number of hyperspheres K_4 containing the five lines $A_1 A_2$, $A_2 A_3$, $A_3 A_4$, $A_4 A_5$, for these lines and an *arbitrary* isotropic line through A_1 lie on a hypersphere K_4. Consequently, the five lines $A_1 A_2$, $A_2 A_3$, ... must lie on a hypersphere K_3.

THEOREM VII. *If a skew pentagon in S_4 is formed of isotropic lines, its sides lie on a hypersphere K_3.*

If we apply Darboux's transformation to the corners of this pentagon, we get five spherical waves touching one another in succession, and the theorem tells us that their points of contact lie on a sphere. It should be remarked that, if five spheres touch one another in succession, the five points of contact do not necessarily lie on a sphere; it is only when an *even* number of the contacts are external contacts that the theorem is true.

§ 5.

A very interesting application of Theorem III may be made to a solution of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

which is a periodic function of t with the period $2T$, and is regular for some domain of values of (x, y, z, t). The integral*

$$U = \int_{t-T}^{t+T} V\,dt$$

is then a solution of $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$, for

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \int_{t-T}^{t+T}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) dt$$

$$= \int_{t-T}^{t+T}\left(\frac{\partial^2 V}{\partial t^2}\right) dt = 0,$$

on account of the periodicity of $\frac{\partial V}{\partial t}$.

Now, Theorem III enables us to express $\int_{t-T}^{t+T} V\,dt$ as an integral taken over the surface of a sphere of radius cT. Hence, we have the theorem that

* A similar theorem is given by Prof. Lamb for the case in which the limits are $-\infty$ and $+\infty$, *Proc. London Math. Soc.*, Vol. VIII, Ser. 2, p. 437.

the mean value of V over a sphere of radius cT is a solution of Laplace's equation, when regarded as a function of the coordinates of the centre of the sphere. It is thought that this theorem may be of fundamental importance in a mathematical theory of gravitation, and that the sphere of radius cT represents the surface of an electron. At present, however, our knowledge is not sufficient to enable us to form any definite theory, and the only use we can make of the theorem is to derive a number of ideas from it and use them as working hypotheses. The ideas which the theorem suggests are:

(1) That the surface of an electron *in motion* can be represented by a sphere in S_4. In the theorem the sphere of integration is situated in a space $t = \text{const.}$, and this sphere may be supposed to represent the surface of an electron at rest. The general form of Theorem III then leads us to consider the mean value of V over an arbitrary sphere in S_4 and a corresponding integral along the line joining the foci.

Let (x_1, y_1, z_1, ict_1), (x_2, y_2, z_2, ict_2), (x, y, z, ict) be the coordinates of the foci and of an arbitrary point on the sphere, and consider the points $F_1(x_1, y_1, z_1)$, $F_2(x_2, y_2, z_2)$, $P(x, y, z)$, which are the centres of the representative waves of these points.

The locus of P is a prolate spheroid having the points F_1 and F_2 as foci. We shall regard P as the position at time t of some point on the surface of an electron. The spheroid then represents a particular *view* of the surface of the electron; it is in fact the view which would be obtained by an observer at one of the foci, *e. g.*, F_2. It should be noticed that different points on the surface of the spheroid have different times associated with them, but this is just what is required in order that disturbances issuing from them should all arrive at F_2 at the same instant. The points of the spheroid will also receive disturbances from F_1 at the same instant.

We shall regard F_1 and F_2 as two positions of the singularity of the electromagnetic field associated with the electron, *i. e.*, of the electromagnetic field considered in § 3. In the case when the spheroid is a sphere, the points F_1 and F_2 coincide and we have supposed that the electron is momentarily at rest. It is possible, however, for the surface to be a sphere when the singularity is describing a periodic orbit with period $2T$. This brings us to the second idea which is suggested by the theorem.

(2) We must look for interpretations of our geometrical theorems in con-

nection with periodic motions. An attempt has already been made to obtain in this way a physical interpretation of Steiner's porisms.*

§ 6.

We shall now prove some fundamental theorems relating to spheres and hyperspheres in S_4 which have a direct bearing upon the kinematics of an electron. According to an idea developed by Minkowski,† Born ‡ and Herglotz,§ the motion of a connected system of particles may be represented by equations of the form

$$\left.\begin{aligned} x &= f_1(X, Y, Z, u), \\ y &= f_2(X, Y, Z, u), \\ z &= f_3(X, Y, Z, u), \\ t &= f_4(X, Y, Z, u), \end{aligned}\right\} \quad (1)$$

where (X, Y, Z) are parameters which enable us to identify a particular particle; they vary from particle to particle, but remain constant during the motion. The different values of u determine different *views* of the connected system, a view being obtained when a time t is associated with a point (x, y, z) according to the law

$$u(x, y, z, t) = \text{const.}$$

We shall now investigate the motion which corresponds to the case in which the loci $u = \text{const.}$ are represented in S_4 by hyperspheres and the equations (1) represent the orthogonal trajectories of these hyperspheres.

Using (x_1, x_2, x_3, x_4) in place of the quantities x, y, z, ict, the equation of a singly infinite system of hyperspheres in S_4 may be written in the form

$$f(u)\left[x_1^2 + x_2^2 + x_3^2 + x_4^2\right] + 2\alpha_1(u)x_1 + 2\alpha_2(u)x_2 + 2\alpha_3(u)x_3 + 2\alpha_4(u)x_4 + 2\phi(u) = 0,$$

the case of a family of hyperplanes being obtained by putting $f(u) \equiv 0$.

---

* *Phys. Zeitschr.* (1910), pp. 318–320. It should be remarked in support of this theory that there are good reasons for supposing that the problem of finding a stable configuration of electrons is poristic, *i. e.*, a configuration will not be stable unless certain conditions (perhaps a single condition) are satisfied, and when these conditions are satisfied an infinite number of configurations will exist. These configurations may be classified into types according to the different ways in which the conditions can be satisfied. Now for each type there will be a group of transformations enabling us to pass from one possible configuration to another, and we should expect this transformation to leave the fundamental equations of electromagnetism unaltered in form. This condition is fulfilled in the case of Steiner's porisms, and in fact in the case of any porism relating to spherical waves in contact, for the conditions of the porism are unaltered by any spherical wave transformation which transforms the fixed spherical waves into themselves.

† *Gött. Nachr.* (1908). *Phys. Zeitschr.* (1909).

‡ *Ann. d. Physik*, Vol. XXX (1909). *Gött. Nachr.* (1909).

§ *Ann. d. Physik*, Vol. XXXI (1910).

Differentiating with regard to x_1, x_2, x_3, x_4, we obtain

$$f x_1 + \alpha_1 + \frac{\partial u}{\partial x_1} M = 0, \quad f x_2 + \alpha_2 + \frac{\partial u}{\partial x_2} M = 0, \quad \text{etc.},$$

where

$$M = \tfrac{1}{2} f'(u) [x_1^2 + x_2^2 + x_3^2 + x_4^2] + \alpha_1'(u) x_1 + \alpha_2'(u) x_2 + \alpha_3'(u) x_3 + \alpha_4'(u) x_4 + \phi'(u).$$

Now, $\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}\right)$ are proportional to the direction cosines of the normal to the hypersphere $u =$ const., and so the differential equations of the orthogonal trajectories are

$$\frac{\frac{d x_1}{d u}}{f x_1 + \alpha_1} = \frac{\frac{d x_2}{d u}}{f x_2 + \alpha_2} = \frac{\frac{d x_3}{d u}}{f x_3 + \alpha_3} = \frac{\frac{d x_4}{d u}}{f x_4 + \alpha_4}.$$

Since

$$(f x_1 + \alpha_1)\frac{d x_1}{d u} + (f x_2 + \alpha_2)\frac{d x_2}{d u} + (f x_3 + \alpha_3)\frac{d x_3}{d u} + (f x_4 + \alpha_4)\frac{d x_4}{d u} + M = 0,$$

it follows that each of the above ratios is equal to

$$-\frac{M}{(f x_1 + \alpha_1)^2 + (f x_2 + \alpha_2)^2 + (f x_3 + \alpha_3)^2 + (f x_4 + \alpha_4)^2} = -\frac{M}{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 - 2 f \phi} = 2 \varkappa M f,$$

where $\varkappa$ is some function of u.

The infinitesimal transformation by which we may pass from a point (x_1, x_2, x_3, x_4) on the hypersphere u to the point (x_1', x_2', x_3', x_4') in which the orthogonal trajectory through (x_1, x_2, x_3, x_4) meets the consecutive hypersphere, is given by

$$x_1' = x_1 + \frac{d x_1}{d u} \delta u, \quad \text{etc.}$$

Substituting the values of $\frac{d x_1}{d u}, \frac{d x_2}{d u}, \frac{d x_3}{d u}, \frac{d x_4}{d u}$, we obtain the relations

$$x_1' = x + 2 \varkappa \delta u (f x_1 + \alpha_1) [(f \alpha_1' - f' \alpha_1) x_1 + (f \alpha_2' - f' \alpha_2) x_2 + (f \alpha_3' - f' \alpha_3) x_3 + (f \alpha_4' - f' \alpha_4) x_4 + f \phi' - f' \phi],$$

where M has been simplified by means of equation (1).

Subtracting the expressions of type

$$\varkappa \delta u (f \alpha_1' - f' \alpha_1) [f (x_1^2 + x_2^2 + x_3^2 + x_4^2) + 2 \alpha_1 x_1 + 2 \alpha_2 x_2 + 2 \alpha_3 x_3 + 2 \alpha_4 x_4 + 2 \phi] \equiv 0$$

from these equations, we obtain the relations

$$\begin{aligned} x_1' = x + \varkappa\,\delta u\,[&f(f\alpha_1' - f'\alpha_1)(x_1^2 - x_2^2 - x_3^2 - x_4^2) \\ &+ 2f(f\alpha_2' - f'\alpha_2)\,x_1 x_2 + 2f(f\alpha_3' - f'\alpha_3)\,x_1 x_3 \\ &+ 2f(f\alpha_4' - f'\alpha_4)\,x_1 x_4 + 2f(f\phi' - f'\phi)\,x_1 \\ &+ 2f(\alpha_1\alpha_2' - \alpha_2\alpha_1')\,x_2 + 2f(\alpha_1\alpha_3' - \alpha_3\alpha_1')\,x_3 \\ &+ 2f(\alpha_1\alpha_4' - \alpha_4\alpha_1')\,x_4 + 2f(\alpha_1\phi' - \phi\alpha_1'), \text{ etc.} \end{aligned}$$

Comparing these with the general equations* of an infinitesimal conformal transformation of S_4, viz.:

$$\begin{aligned} x_1' &= x_1 + \varepsilon\,[p\,(x_1^2 - x_2^2 - x_3^2 - x_4^2) + 2\,q\,x_1 x_2 + 2\,r\,x_1 x_3 + 2\,s\,x_1 x_4 \\ &\qquad + \mu\,x_1 + h\,x_2 + g\,x_4 + l\,x_4 + a], \\ x_2' &= x_2 + \varepsilon\,[q\,(x_2^2 - x_3^2 - x_4^2 - x_1^2) + 2\,p\,x_1 x_2 + 2\,r\,x_2 x_3 + 2\,s\,x_2 x_4 \\ &\qquad - h\,x_1 + \mu\,x_2 + f\,x_3 + m\,x_4 + b], \\ x_3' &= x_3 + \varepsilon\,[r\,(x_3^2 - x_4^2 - x_1^2 - x_2^2) + 2\,p\,x_1 x_3 + 2\,q\,x_2 x_3 + 2\,s\,x_3 x_4 \\ &\qquad - g\,x_1 - f\,x_2 + \mu\,x_3 + n\,x_4 + c], \\ x_4' &= x_4 + \varepsilon\,[s\,(x_4^2 - x_1^2 - x_2^2 - x_3^2) + 2\,p\,x_1 x_4 + 2\,q\,x_2 x_4 + 2\,r\,x_3 x_4 \\ &\qquad - l\,x_1 - m\,x_2 - n\,x_3 - \mu\,x_4 + d], \end{aligned}$$

we see that the transformation from one hypersphere to the next can be effected by means of an infinitesimal conformal transformation of S_4.†

This result is important because we know that a conformal transformation of S_4 transforms a sphere into a sphere and a circle into a circle. Hence:

THEOREM VIII. *If a tube of orthogonal trajectories of the hyperspheres $u =$ const. cuts one hypersphere in a sphere, it will cut every hypersphere in a sphere; if it cuts one hypersphere in a circle, it will cut every hypersphere in a circle; if it cuts one hypersphere in an isotropic line, it will cut every hypersphere in an isotropic line.* ‡

It follows from this theorem that, if the points of a material system move in such a way that their paths are represented in S_4 by the orthogonal trajectories of the hyperspheres $u =$ const., then, if the view determined by one constant value of u is such that the material system is represented by spheres or circles in S_4, then the material system is represented by spheres or circles in S_4 in any other view given by an equation $u =$ const.

---

* See p. 243 of my paper on the transformation of the electrodynamical equations, *Proc. London Math. Soc.*, Vol. VIII (1910), Ser. 2, Part 4.

† This theorem is practically proved in Darboux's Leçons sur les systèmes orthogonaux, t. I (1898), Ch. II.

‡ There is a similar theorem for spheres, circles and isotropic lines in a space of three dimensions.

In the particular case when the loci $u =$ const. are hyperplanes, the figures which are cut out on each by a family of orthogonal trajectories may be derived from one another by displacements in S_4. This is the case considered by Born and Herglotz.

The case when the loci $u =$ const are isotropic cones (*i. e.*, hyperspheres of zero radius) is particularly interesting, for then the tangent to an orthogonal trajectory at a point P is also a generator to the isotropic cone through P. For, if the isotropic cones are

$$[x_1 - \xi_1(u)]^2 + [x_2 - \xi_2(u)]^2 + [x_3 - \xi_3(u)]^2 + [x_4 - \xi_4(u)]^2 = 0,$$

the orthogonal trajectories are

$$\frac{d x_1}{x_1 - \xi_1} = \frac{d x_2}{x_2 - \xi_2} = \frac{d x_3}{x_3 - \xi_3} = \frac{d x_4}{x_4 - \xi_4}.$$

Putting $\qquad x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = i c t,$

$$\xi_1(u) = \xi(\tau), \quad \xi_2(u) = \eta(\tau), \quad \xi_3(u) = \zeta(\tau), \quad \xi_4(u) = \tau,$$

we obtain the equations

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2 [t - \tau]^2,$$

$$\frac{d x}{x - \xi} = \frac{d y}{y - \eta} = \frac{d z}{z - \zeta} = \frac{d t}{t - \tau}.$$

The first equation shows that a particle at (x, y, z) at time t is active relative to (ξ, η, ζ, τ), or vice versa, and the second equation shows that it is moving with the velocity of light along the line joining the two particle.

Theorem VIII may be generalized by considering the orthogonal trajectories of a singly infinite family of spheres in S_4. We shall show that the orthogonal trajectories map the spheres on one another in such a way that the transition from a figure on one sphere to the corresponding figure on another may be effected by means of a conformal transformation of the space S_4.

Let the axes of coordinates be chosen so that the equations of one of the spheres are $\qquad x_1^2 + x_2^2 + x_3^2 = a^2, \quad x_4 = 0.$

Let a consecutive point on the orthogonal trajectory through the point $(a l_1, a l_2, a l_3)$ be given by the equations

$$x_1' = l_1(a + \delta u), \qquad x_3' = l_3(a + \delta u),$$
$$x_2' = l_2(a + \delta u), \qquad x_4' = \theta\, \delta u,$$

and let the equations of the consecutive sphere be

$$(x_1 + \delta \alpha_1)^2 + (x_2 + \delta \alpha_2)^2 + (x_3 + \delta \alpha_3)^2 + (x_4 + \delta \alpha_4)^2 = (a + \delta a)^2,$$
$$x_1 \delta \xi_1 + x_2 \delta \xi_2 + x_3 \delta \xi_3 + x_4 (1 + \delta \rho) = \delta p.$$

Since these equations are satisfied by (x_1', x_2', x_3', x_4'), we have

$$l_1(l_1\delta u+\delta\alpha_1)+l_2(l_2\delta u+\delta\alpha_2)+l_3(l_3\delta u+\delta\alpha_3)=\delta a,$$
$$a\,[l_1\delta\xi_1+l_2\delta\xi_2+l_3\delta\xi_3]+\theta\,\delta u=\delta p,$$

where quantities of the first order only are retained. Since $l_1^2+l_2^2+l_3^2=1$, the first equation gives

$$\delta u=\delta a-l_1\delta\alpha_1-l_2\delta\alpha_2-l_3\delta\alpha_3.$$

The equations of transformation may now be written in the form

$$x_1'=x_1+l_1(\delta a-l_1\delta\alpha_1-l_2\delta\alpha_2-l_3\delta\alpha_3)+\frac{\delta\alpha_1}{2a^2}[x_1^2+x_2^2+x_3^2+x_4^2-a^2],$$
$$x_2'=x_2+l_2(\delta a-l_1\delta\alpha_1-l_2\delta\alpha_2-l_3\delta\alpha_3)+\frac{\delta\alpha_2}{2a^2}[x_1^2+x_2^2+x_3^2+x_4^2-a^2],$$
$$x_3'=x_3+l_3(\delta a-l_1\delta\alpha_1-l_2\delta\alpha_2-l_3\delta\alpha_3)+\frac{\delta\alpha_3}{2a^2}[x_1^2+x_2^2+x_3^2+x_4^2-a^2],$$
$$x_4'=\delta p-a\,l_1\delta\xi_1-a\,l_2\delta\xi_2-a\,l_3\delta\xi_3+\frac{\delta a}{a}x_4+x_4,$$

where multiples of the zero quantities $x_1^2+x_2^2+x_3^2+x_4^2-a^2$, x_4 have been added.

Putting $a\,l_1=x_1$, $a\,l_2=x_2$, $a\,l_3=x_3$, we have

$$x_1'=x_1+\frac{\delta\alpha_1}{2a^2}(x_2^2+x_3^2+x_4^2-x_1^2)-\frac{\delta\alpha_2}{a^2}x_1x_2-\frac{\delta\alpha_3}{a^2}x_1x_3+\frac{\delta a}{a}x_1+x_4\delta\xi_1-\frac{\delta\alpha_1}{2},$$
$$x_2'=x_2+\frac{\delta\alpha_2}{2a^2}(x_3^2+x_4^2+x_1^2-x_2^2)-\frac{\delta\alpha_1}{a^2}x_1x_2-\frac{\delta\alpha_3}{a^2}x_2x_3+\frac{\delta a}{a}x_2+x_4\delta\xi_2-\frac{\delta\alpha_2}{2},$$
$$x_3'=x_3+\frac{\delta\alpha_3}{2a^2}(x_4^2+x_1^2+x_2^2-x_3^2)-\frac{\delta\alpha_1}{a^2}x_1x_3-\frac{\delta\alpha_2}{a^2}x_2x_3+\frac{\delta a}{a}x_3+x_4\delta\xi_3-\frac{\delta\alpha_3}{2},$$
$$x_4'=x_4-x_1\delta\xi_1-x_2\delta\xi_2-x_3\delta\xi_3+x_4\frac{\delta a}{a}.$$

Now, $\delta\alpha_1$, $\delta\alpha_2$, $\delta\alpha_3$, $\delta\xi_1$, $\delta\xi_2$, $\delta\xi_3$, δa depend only on the two spheres and not on the coordinates of a particular point (x_1, x_2, x_3, x_4) on the first sphere; hence they may be regarded as constants as far as the transformation is concerned. This being the case, it is easy to see that the above equations represent an infinitesimal conformal transformation of the space S_4, and since such a transformation transforms a circle into a circle, we have the following theorem:

THEOREM IX. *The orthogonal trajectories of a family of ∞^1 spheres in a space S_4 map the spheres conformally on one another in such a way that a circle is always mapped on a circle and an isotropic line on an isotropic line.*

These theorems are of considerable interest in connection with the problem of finding solutions of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t_2},$$

having the general form

$$V = W F(X, Y, Z),$$

where F is any solution of a partial differential equation of the second order in the independent variables X, Y, Z.

I have shown elsewhere* that this problem can be reduced to that of determining a transformation

$$X = f_1(x_1, x_2, x_3, x_4), \quad Y = f_2(x_1, x_2, x_3, x_4), \quad Z = f_3(x_1, x_2, x_3, x_4),$$

such that there is an identical relation of the form

$$(p_1^2 + p_2^2 + p_3^2 + p_4^2)(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) \\ - (p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4)^2 \equiv l dX^2 + m dY^2 + n dZ^2, \quad (1)$$

where l, m, n are functions of X, Y, Z only, and x_1, x_2, x_3, x_4 are written instead of x, y, z, ict.

I have also shown that the necessary and sufficient conditions for the existence of a relation of this type are that the following set of equations should be satisfied.

$$\left.\begin{aligned}
&\frac{\partial p_2}{\partial x_3} + \frac{\partial p_3}{\partial x_2} = \xi_3 p_2 + \xi_2 p_3, \quad \frac{\partial p_4}{\partial x_1} + \frac{\partial p_1}{\partial x_4} = \xi_4 p_1 + \xi_1 p_4, \\
&\frac{\partial p_3}{\partial x_1} + \frac{\partial p_1}{\partial x_3} = \xi_1 p_3 + \xi_3 p_1, \quad \frac{\partial p_4}{\partial x_2} + \frac{\partial p_2}{\partial x_4} = \xi_4 p_2 + \xi_2 p_4, \\
&\frac{\partial p_1}{\partial x_2} + \frac{\partial p_2}{\partial x_1} = \xi_2 p_1 + \xi_1 p_2, \quad \frac{\partial p_4}{\partial x_3} + \frac{\partial p_3}{\partial x_4} = \xi_4 p_3 + \xi_3 p_4, \\
&\qquad \frac{\partial p_1}{\partial x_1} - \frac{\partial p_2}{\partial x_2} - \frac{\partial p_3}{\partial x_3} - \frac{\partial p_4}{\partial x_4} = 2\xi_1 p_1, \\
&\qquad \frac{\partial p_2}{\partial x_2} - \frac{\partial p_3}{\partial x_3} - \frac{\partial p_4}{\partial x_4} - \frac{\partial p_1}{\partial x_1} = 2\xi_2 p_2, \\
&\qquad \frac{\partial p_3}{\partial x_3} - \frac{\partial p_4}{\partial x_4} - \frac{\partial p_1}{\partial x_1} - \frac{\partial p_2}{\partial x_2} = 2\xi_3 p_3, \\
&\qquad \frac{\partial p_4}{\partial x_4} - \frac{\partial p_1}{\partial x_1} - \frac{\partial p_2}{\partial x_2} - \frac{\partial p_3}{\partial x_3} = 2\xi_4 p_4,
\end{aligned}\right\} \quad (2)$$

where $\xi_1, \xi_2, \xi_3, \xi_4$ are certain functions defined by the last four equations.

* *Cambr. Phil. Trans.*, Vol. XXI (1910), pp. 257–280.

It should be remarked that these equations give the solution of the more general problem of finding solutions of

$$\frac{\partial}{\partial x}\left(K\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial V}{\partial z}\right)=\frac{1}{c^2}\frac{\partial}{\partial t}\left(K\frac{\partial V}{\partial t}\right) \tag{3}$$

which have the required form. It does not follow that, when the conditions are satisfied, the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \tag{4}$$

possesses solutions of the required type, but if it does possess solutions of the required type, the conditions must be satisfied, and so in searching for these solutions we must obtain the general solution of the differential equations (2). It should be remarked that the equation (3) possesses the same characteristics as (4), and so also corresponds to a type of wave motion in which disturbances are propagated with the velocity of light.

We shall now obtain the solution of the equations (2) in two cases, first when $p_1dx_1+p_2dx_2+p_3dx_3+p_4dx_4$ can be reduced to the form Udu, and secondly, when it can be reduced to the form $Udu+Vdv$. These are the only cases which need be considered.

Using suffixes to denote differentiations with regard to the variables x_1, x_2, x_3, x_4, we have in the first case

$$p_1=Uu_1,\quad p_2=Uu_2,\quad p_3=Uu_3,\quad p_4=Uu_4.$$

The equations (2) may be written in the form

$$2\,Uu_{23}=(U\xi_3-U_3)\,u_2+(U\xi_2-U_2)\,u_3,$$
$$\dots\dots\dots\dots\dots\dots,$$
$$U(u_{11}-u_{22}-u_{33}-u_{44})=2\,(U\xi_1-U_1)\,u_1+(U_1u_1+U_2u_2+U_3u_3+U_4u_4),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

and imply that there is an identical relation of the form

$$\begin{aligned}2\,U[&u_{11}dx_1^2+u_{22}dx_2^2+u_{33}dx_3^2+u_{44}dx_4^2+2\,u_{23}dx_2dx_3+2\,u_{31}dx_3dx_1\\&+2\,u_{12}dx_1dx_2+2\,u_{14}dx_1dx_4+2\,u_{24}dx_2dx_4+2\,u_{34}dx_3dx_4]\\\equiv{}&[dx_1^2+dx_2^2+dx_3^2+dx_4^2]\,[U(u_{11}+u_{21}+u_{33}+u_{44})\\&+(U_1u_1+U_2u_2+U_3u_3+U_4u_4)]+2\,[(U\xi_1-U_1)\,dx_1\\&+(U\xi_2-U_2)\,dx_2+(U\xi_3-U_3)\,dx_3+(U\xi_4-U_4)\,dx_4]\\&[u_1dx_1+u_2dx_2+u_3dx_3+u_4dx_4].\end{aligned}$$

Regarding (dx_1, dx_2, dx_3, dx_4) as current coordinates in the neighborhood of a point (x_1, x_2, x_3, x_4) on the manifold $u = \text{const.}$, we see from the above equation that the indicatrix (*i. e.*, the section by a hyperplane parallel to $u_1 dx_1 + u_2 dx_2 + u_3 dx_3 + u_4 dx_4 = 0$) is a sphere at all points of the hypersurface. Now this property is characteristic of a hypersphere and also of a type of isotropic developable. The latter need not be taken into consideration, because it is specified by an equation of the type

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0,$$

and this would give $p_1^2 + p_2^2 + p_3^2 + p_4^2 = 0$. Hence we may conclude that the hypersurfaces $u = \text{const.}$ are hyperspheres. Conversely, it may be shown that if the manifolds $u = \text{const.}$ are hyperspheres, there is an identical relation of the form

$$\begin{aligned}(p_1^2 + p_2^2 + p_3^2 + p_4^2)(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) \\ - (p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4)^2 = l\,dX^2 + m\,dY^2 + n\,dZ^2.\end{aligned}$$

For, it is known that any family of hyperspheres forms part of a quadruply infinite system of manifolds.* Let these be $u = \text{const.}$, $X = \text{const.}$, $Y = \text{const.}$, $Z = \text{const.}$; then there will be a relation of the form

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = \lambda\,du^2 + \mu\,[l\,dX^2 + m\,dY^2 + n\,dZ^2].$$

Since, moreover, the orthogonal trajectories map the manifolds $u = \text{const.}$ conformally on one another, we can choose μ so that l, m, n are independent of u. Dividing out by μ, the equation takes the form

$$\begin{aligned}\frac{1}{\mu}[dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2] - \frac{\lambda}{\mu}[u_1 dx_1 + u_2 dx_2 + u_3 dx_3 + u_4 dx_4]^2 \\ = l\,dX^2 + m\,dY^2 + n\,dZ^2,\end{aligned}$$

and since the Jacobian of the quadratic form on the left is zero, the quantity on the left-hand side must be expressible in the form

$$\begin{aligned}(p_1^2 + p_2^2 + p_3^2 + p_4^2)(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) \\ - (p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4)^2.\end{aligned}$$

This proves the theorem.

In the second case, when

$$p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4 = U\,du + V\,dv,$$

---

* Cf. Darboux, Leçons sur les systèmes orthogonaux, Ch. II.

the equations (2) become

$$2Uu_{23}+2Vv_{23}=(U\xi_3-U_3)u_2+(U\xi_2-U_2)u_3+(V\xi_3-V_3)v_2+(V\xi_2-V_2)v_3,$$
$$\begin{aligned}&U(u_{11}-u_{22}-u_{33}-u_{44})+V(v_{11}-v_{22}-v_{33}-v_{44})\\&\quad=2(U\xi_1-U_1)u_1+2(V\xi_1-V_1)v_1\\&\qquad+(U_1u_1+U_2u_2+U_3u_3+U_4u_4)+(V_1v_1+V_2v_2+V_3v_3+V_4v_4),\end{aligned}$$

and these imply that there is an identical relation of the form

$$\begin{aligned}&2U[u_{11}dx_1^2+\dots\dots]+2V[v_{11}dx_1^2+\dots\dots]\\&\quad=[dx_1^2+dx_2^2+dx_3^2+dx_4^2][U(u_{11}+u_{22}+u_{33}+u_{44})+V(v_{11}+v_{22}+v_{33}+v_{44})\\&\qquad+(U_1u_1+U_2u_2+U_3u_3+U_4u_4)+(V_1v_1+V_2v_2+V_3v_3+V_4v_4)]\\&\qquad+2[(U\xi_1-U_1)dx_1+(U\xi_2-U_2)dx_2+(U\xi_3-U_3)dx_3+(U\xi_4-U_4)dx_4]\\&\qquad\quad[u_1dx_1+u_2dx_2+u_3dx_3+u_4dx_4]\\&\qquad+2[(V\xi_1-V_1)dx_1+(V\xi_2-V_2)dx_2+(V\xi_3-V_3)dx_3+(V\xi_4-V_4)dx_4]\\&\qquad\quad[v_1dx_1+v_2dx_2+v_3dx_3+v_4dx_4].\end{aligned}$$

Regarding (dx_1, dx_2, dx_3, dx_4) as current coordinates of a point in the neighborhood of the point (x_1, x_2, x_3, x_4) on the surface $u=$ const., $v=$ const., we see that the indicatrix of this surface, *i. e.*, the section by a plane parallel to the tangent plane

$$u_1dx_1+u_2dx_2+u_3dx_3+u_4dx_4=0,$$
$$v_1dx_1+v_2dx_2+v_3dx_3+v_4dx_4=0,$$

is a *circle* at every point of the surface. Hence, we may conclude that the surface is a *sphere.* This gives the following theorem:

If there is an identical relation of the form (1), *where*

$$p_1dx_1+p_2dx_2+p_3dx_3+p_4dx_4=Udu+Vdv,$$

the manifolds $u=$ const., $v=$ const. intersect in a sphere or a number of spheres.

In the motion represented by the orthogonal trajectories of a family of hyperspheres or spheres in S_4, the component velocities (w_x, w_y, w_z) of the particle which is at the point (x, y, z) at time t and is represented by (x_1, x_2, x_3, x_4) in S_4, are given by the equations

$$\frac{w_x}{p_1}=\frac{w_x}{p_2}=\frac{w_x}{p_3}=\frac{ic}{p_4},$$

and so the relation

$$p_1^2+p_2^2+p_3^2+p_4^2\neq 0$$

implies that $w_x^2+w_y^2+w_z^2 \neq c^2$, or that the velocity of the particle is not equal to the velocity of light. It was shown in my former paper that there is an integral invariant of the form

$$(p_1^2+p_2^2+p_3^2+p_4^2)\,[p_1\,dx_2\,dx_3\,dx_4+p_2\,dx_3\,dx_1\,dx_4+p_3\,dx_1\,dx_2\,dx_4 \\ -p_4\,dx_1\,dx_2\,dx_3] \equiv \sqrt{lmn}\,dX\,dY\,dZ,$$

or
$$\rho\,[w_x\,dy\,dz\,dt+w_y\,dz\,dx\,dt+w_z\,dx\,dy\,dt-dx\,dy\,dz] \\ = \sqrt{lmn}\,dX\,dY\,dZ.$$

The interpretation of this equation is that the *electric charge** associated with a number of particles remains constant during the motion, *i. e.*, as we pass from one view to another. In particular, if an electron is represented by a sphere in S_4, the electric charge associated with the electron will remain constant.

Again, the equation

$$\frac{\partial}{\partial x_1}[p_1(p_1^2+p_2^2+p_3^2+p_4^2)]+\frac{\partial}{\partial x_2}[p_2(p_1^2+p_2^2+p_3^2+p_4^2)] \\ +\frac{\partial}{\partial x_3}[p_3(p_1^2+p_2^2+p_3^2+p_4^2)]+\frac{\partial}{\partial x_4}[p_4(p_1^2+p_2^2+p_3^2+p_4^2)]=0,$$

which was obtained on p. 269 of my former paper, takes the form

$$\frac{\partial}{\partial x}(\rho w_x)+\frac{\partial}{\partial y}(\rho w_y)+\frac{\partial}{\partial z}(\rho w_z)+\frac{\partial \rho}{\partial t}=0,$$

and implies that the equation of continuity is satisfied.

§ 7.

It should be remarked that there is a similar set of theorems connected with Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

and the more general equation

$$\frac{\partial}{\partial x}\left(K\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial V}{\partial z}\right)=0.$$

---

* See p. 229 of my paper on the transformation of the electrodynamical equations, *Proc. London Math. Soc.*, Vol. VIII (1910).

If we seek a set of solutions of the form*

$$V = Wf(X, Y),$$

we find that there must be an identical relation of the form

$$(p^2+q^2+r^2)(dx^2+dy^2+dz^2)-(p\,dx+q\,dy+r\,dz)^2 = a\,dX^2+b\,dY^2+2h\,dX\,dY,$$

where a, b, h are functions of X and Y. The functions p, q, r are determined by differential equations of the form

$$\frac{\partial Q}{\partial z}+\frac{\partial R}{\partial y}=\zeta Q+\eta R, \qquad \frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}-\frac{\partial R}{\partial z}=2\xi P,$$

$$\frac{\partial R}{\partial x}+\frac{\partial P}{\partial z}=\xi R+\zeta P, \qquad \frac{\partial Q}{\partial y}-\frac{\partial R}{\partial z}-\frac{\partial P}{\partial x}=2\eta Q,$$

$$\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}=\eta P+\xi Q, \qquad \frac{\partial R}{\partial z}-\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}=2\zeta R,$$

where

$$P=\frac{p}{\sqrt{p^2+q^2+r^2}}, \quad Q=\frac{q}{\sqrt{p^2+q^2+r^2}}, \quad R=\frac{r}{\sqrt{p^2+q^2+r^2}},$$

and ξ, η, ζ are certain functions which may be defined by means of three of these equations.

Since any quadratic form in two variables, whose discriminant is not zero, may be reduced to the form $\lambda\,[d\bar{X}^2+d\,\bar{Y}^2]$, we see that the above equations represent the necessary and sufficient conditions that there should be an identity of the form

$$(p^2+q^2+r^2)(dx^2+dy^2+dz^2)-(p\,dx+q\,dy+r\,dz)^2=\lambda\,[d\bar{X}^2+d\,Y^2].$$

It can now be shown in the same way as in § 6 that, if $p\,dx+q\,dy+r\,dz$ can be reduced to the form $U\,du$, the surfaces $u=$ const. are spheres, and if it can be reduced to the form $U\,du+V\,dv$, the surfaces $u=$ const., $v=$ const. intersect in circles. Also we have the following theorems, both of which are well-known:

The orthogonal trajectories of a family of ∞^1 spheres map the spheres conformally on one another, a circle being mapped into a circle and an isotropic line into an isotropic line.

---

* These solutions have been considered from a different point of view by U. Amaldi, *Rend. Palermo*, Vol. XVI (1902), p. 1, and less generally by Levi Civita, *Mem. dell' Acc. di Torino*, s. II, t. XLIX (1899).

The orthogonal trajectories of a family of ∞^1 circles map the circles conformally on one another; i. e., the cross-ratio of four points on one circle is equal to the cross-ratio of four corresponding points on another.

The last theorem occurs in the theory of cyclic systems, and the present theory shows that the theory of cyclic systems should be connected with that of Laplace's equation. We shall not investigate this relation in the present paper, but it may be useful to state the theorem corresponding to the case when $p\,dx + q\,dy + r\,dz = U\,du$, and $u =$ const. represents a family of spheres.

THEOREM. *If u is defined by the equation*

$$x^2 + y^2 + z^2 + 2\alpha(u)x + 2\beta(u)y + 2\gamma(u)z + 2\phi(u) = 0,$$

then

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right](dx^2 + dy^2 + dz^2) - \left[\frac{\partial u}{\partial x}dx + \frac{du}{\partial y}dy + \frac{\partial u}{\partial z}\partial z\right]^2$$

can be reduced to the form $\lambda(dX^2 + dY^2)$, and any function F which satisfies the equation

$$\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = 0$$

is a solution of the two equations

$$\frac{\partial}{\partial x}\left(K\frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial F}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial F}{\partial z}\right) = 0,$$

$$\frac{\partial u}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial F}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial F}{\partial z} = 0,$$

where

$$K = \frac{\chi(u)}{x\alpha'(u) + y\beta'(u) + z\gamma'(u) + \phi'(u)},$$

and $\chi(u)$ is an arbitrary function. If $\chi(u)$ can be chosen so that $\sqrt{K}$ is a solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$, then $V = \sqrt{K}.F(X, Y)$ will also be a solution of this equation.

§ 8.

The group G_{15} of conformal transformations of the space S_4, which is equivalent to the group of spherical wave transformations, is composed of displacements, reflexions and inversions in the space S_4 and of transformations

which may be obtained by compounding these. Thus magnifications and rotations enter as particular transformations of the group. The rotations are of great importance, because, as Minkowski has shown, Lorentz's transformation *

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad z' = z, \quad y' = y$$

corresponds to a rotation

$$\begin{aligned} x_1' &= x_1 \cos\omega + x_4 \sin\omega, & x_2' &= x_2, \\ x_4' &= x_4 \cos\omega - x_1 \sin\omega, & x_3' &= x_3, \end{aligned} \qquad \tan\omega = \frac{iv}{c}$$

in the four-dimensional space S_4.

When we use Darboux's representation of a point in S_4 by a spherical wave in S_3, the group G_{15} becomes the group of spherical wave transformations which transform a spherical wave into a spherical wave. This group of transformations has been discussed by S. Lie; † it is the group of transformations which transform lines of curvature on a surface enveloped by spherical waves into lines of curvature on the surface enveloped by the corresponding spherical waves.

An inversion of one set of spherical waves into another is a particular instance of a spherical wave transformation, and this corresponds to an inversion in S_4. A dilatation in which the radii of all the spherical waves are increased or diminished by the same amount is another example, and this corresponds to a displacement parallel to the axis of x_4 in the space S_4.

Ribaucour's transformation, which transforms lines of curvature into lines of curvature, is easily seen to correspond to Lorentz's transformation, for, if we take the equations given on p. 254 of Darboux's "Théorie des Surfaces," t. 1, viz.,

$$\begin{aligned} x' &= x, & z' &= \frac{1+k^2}{1-k^2} z - \frac{2kR}{1-k^2}, \\ y' &= y, & R' &= \frac{2kz}{1-k^2} - \frac{1+k^2}{1-k^2} R, \end{aligned}$$

---

* *Amsterdam Proceedings*, 1903–04.

† *Gött. Nachr.* (1871). "Transformationsgruppen," Bd. 3, p. 351. "Infinitesimale Berührungstransformationen der Optik." *Leipziger Berichte*, Math.-Phys. Classe, Bd. 48 (1896), pp. 131–133. See also papers by E. O. Lovett, "Contact Transformations and Optics," *Cambr. Phil. Soc.*, Commemoration Volume (1900), p. 256, and P. F. Smith, *Trans. Amer. Math. Soc.*, Vol. 1 (1900), p. 371.

and write ct, ct' in place of R and R', and $\frac{v}{c}$ in place of $\frac{2k}{1-k^2}$, we obtain at once a form of Lorentz's transformation which is appropriate for the case of a uniform motion parallel to the axis of z.

On account of the importance of the transformation theory in electrodynamics, it is desirable to study the group G_{15} of spherical wave transformations in full detail. This may be done very conveniently by means of the following theorem, which is due to Klein:*

The general conformal group in four dimensions is simply isomorphic with the general projective group in three.

This theorem depends on Lie's correspondence between a line and a spherical wave. A projective transformation which transforms a line into a line corresponds to a spherical wave transformation which transforms a spherical wave into a spherical wave.

Spherical wave transformations may thus be studied in connection with projective transformations in S_3. The following examples will illustrate this:

(1) In the general projective transformation in S_3 there are four points and six lines joining them which are transformed into themselves. Now a point in S_3 corresponds to a minimal line. Hence, in the general spherical wave transformation there are four isotropic lines which are transformed into themselves; the six spheres containing these lines in pairs are also transformed into themselves. Each of these spheres touches four of the others.

(2) In a homology in S_3 there is a point (the centre of homology) and a plane (the plane of homology) such that every line through the point and every line in the plane is transformed into itself. In the corresponding spherical wave transformation there are two isotropic lines such that any spherical wave through one of them is transformed into itself.

(3) In a skew reflexion the line joining two corresponding points P, Q meets two fixed lines r, s, and is divided harmonically by them. The line PQ is clearly transformed into itself. In the corresponding spherical wave transformation there are two spherical waves R, S which are transformed into themselves, and every wave touchig R and S is transformed into itself.

(4) If a, b, c, d are four non-intersecting straight lines, a projective transformation which will interchange a and c, b and d may be obtained in the

* *Höhere Geometrie*, Vol. I, p. 487, sqq. See also a paper by J. E. Wright, *Trans. Amer. Math. Soc.* (1906).

following way: Draw lines PAB, PCD through an arbitrary point P to meet a, b; c, d in A, B; C, D respectively. Let AD, BC meet in Q; then Q is the point corresponding to P. It is clear that when P is on a, Q will be on c, and when P is on b, Q will be on d. If the common transversals of the four lines a, b, c, d meet them in $\alpha, \beta, \gamma, \delta$; $\alpha', \beta', \gamma', \delta'$, and points X, Y; Z, W are taken on $(\alpha\beta\gamma\delta)$, $(\alpha'\beta'\gamma'\delta')$ respectively, so that $(\alpha\gamma XY)$, $(\beta\delta XY)$; $(\alpha'\gamma' ZW)$, $(\beta'\delta' ZW)$ are harmonic ranges, then X, Y, Z, W are the fixed points of this projective transformation.

The corresponding spherical wave transformation is such that a spherical wave a is transformed into c and another wave b into d. This transformation shows that when we are dealing with the geometrical properties of four spherical waves, the waves may be interchanged in pairs and a corresponding property will still be possessed by the system, provided the property is covariantive for the group of spherical wave transformations.

BRYN MAWR, *December* 24, 1910.